

$$\int \frac{x (A + B \sinh[c + d x])}{(a + b \sinh[c + d x])^2} dx$$

■ **Derivation:** Integration by parts

■ **Rule:** If  $a A + b B = 0$ , then

$$\int \frac{x (A + B \sinh[c + d x])}{(a + b \sinh[c + d x])^2} dx \rightarrow \frac{B x \cosh[c + d x]}{a d (a + b \sinh[c + d x])} - \frac{B}{a d} \int \frac{\cosh[c + d x]}{a + b \sinh[c + d x]} dx$$

■ **Program code:**

```
Int[x*(A+B_.*Sinh[c_.+d_.*x_])/(a_+b_.*Sinh[c_.+d_.*x_])^2,x_Symbol] :=
  B*x*Cosh[c+d*x]/(a*d*(a+b*Sinh[c+d*x])) -
  Dist[B/(a*d),Int[Cosh[c+d*x]/(a+b*Sinh[c+d*x]),x]] /;
FreeQ[{a,b,c,d,A,B},x] && ZeroQ[a*A+b*B]
```

```
Int[x*(A+B_.*Cosh[c_.+d_.*x_])/(a_+b_.*Cosh[c_.+d_.*x_])^2,x_Symbol] :=
  B*x*Sinh[c+d*x]/(a*d*(a+b*Cosh[c+d*x])) -
  Dist[B/(a*d),Int[Sinh[c+d*x]/(a+b*Cosh[c+d*x]),x]] /;
FreeQ[{a,b,c,d,A,B},x] && ZeroQ[a*A-b*B]
```

$$\int \sinh[a + b x]^m \tanh[a + b x]^n dx$$

■ Reference: G&R 2.423.18'

■ Derivation: Algebraic expansion

■ Basis:  $\sinh[z] \tanh[z] = \cosh[z] - \operatorname{sech}[z]$

■ Rule:

$$\int \sinh[a + b x] \tanh[a + b x] dx \rightarrow \frac{\sinh[a + b x]}{b} - \int \operatorname{sech}[a + b x] dx$$

■ Program code:

```
Int[Sinh[a_.+b_.*x_]*Tanh[a_.+b_.*x_],x_Symbol] :=
  Sinh[a+b*x]/b - Int[Sech[a+b*x],x] /;
FreeQ[{a,b},x]
```

■ Reference: G&R 2.423.34'

```
Int[Cosh[a_.+b_.*x_]*Coth[a_.+b_.*x_],x_Symbol] :=
  Cosh[a+b*x]/b + Int[Csch[a+b*x],x] /;
FreeQ[{a,b},x]
```

■ Rule: If  $m + n - 1 = 0$ , then

$$\int \sinh[a + b x]^m \tanh[a + b x]^n dx \rightarrow \frac{\sinh[a + b x]^m \tanh[a + b x]^{n-1}}{b m}$$

■ Program code:

```
Int[Sinh[a_.+b_.*x_]^m_*Tanh[a_.+b_.*x_]^n_,x_Symbol] :=
  Sinh[a+b*x]^m*Tanh[a+b*x]^(n-1)/(b*m) /;
FreeQ[{a,b,m,n},x] && ZeroQ[m+n-1]
```

```
Int[Cosh[a_.+b_.*x_]^m_*Coth[a_.+b_.*x_]^n_,x_Symbol] :=
  Cosh[a+b*x]^m*Coth[a+b*x]^(n-1)/(b*m) /;
FreeQ[{a,b,m,n},x] && ZeroQ[m+n-1]
```

■ **Derivation: Integration by substitution**

■ **Basis:** If  $m, n, \frac{m+n-1}{2} \in \mathbb{Z}$ , then  $\text{Sinh}[z]^m \text{Tanh}[z]^n = \frac{(-1+\text{Cosh}[z]^2)^{\frac{m+n-1}{2}}}{\text{Cosh}[z]^n} \partial_z \text{Cosh}[z]$

■ **Note:** This rule is used if  $m+n$  is odd since it requires fewer steps and results in a simpler antiderivative than the other rules in this section.

■ **Rule:** If  $m, n, \frac{m+n-1}{2} \in \mathbb{Z}$ , then

$$\int \text{Sinh}[a+bx]^m \text{Tanh}[a+bx]^n dx \rightarrow \frac{1}{b} \text{Subst} \left[ \int \frac{(-1+x^2)^{\frac{m+n-1}{2}}}{x^n} dx, x, \text{Cosh}[a+bx] \right]$$

■ **Program code:**

```
Int[Sinh[a_.+b_.*x_]^m_.*Tanh[a_.+b_.*x_]^n_,x_Symbol] :=
  Dist[1/b,Subst[Int[Regularize[(-1+x^2)^(m+n-1)/2]/x^n,x],x],x,Cosh[a+b*x]] /;
FreeQ[{a,b},x] && IntegersQ[m,n,(m+n-1)/2]
```

■ **Basis:** If  $m, n, \frac{m+n-1}{2} \in \mathbb{Z}$ , then  $\text{Cosh}[z]^m \text{Coth}[z]^n = \frac{(1+\text{Sinh}[z]^2)^{\frac{m+n-1}{2}}}{\text{Sinh}[z]^n} \partial_z \text{Sinh}[z]$

```
Int[Cosh[a_.+b_.*x_]^m_.*Coth[a_.+b_.*x_]^n_,x_Symbol] :=
  Dist[1/b,Subst[Int[Regularize[(1+x^2)^(m+n-1)/2]/x^n,x],x],x,Sinh[a+b*x]] /;
FreeQ[{a,b},x] && IntegersQ[m,n,(m+n-1)/2]
```

■ **Reference:** G&R 2.411.1, CRC 567a

■ **Rule:** If  $m > 1 \wedge n < -1$ , then

$$\int \text{Sinh}[a+bx]^m \text{Tanh}[a+bx]^n dx \rightarrow \frac{\text{Sinh}[a+bx]^m \text{Tanh}[a+bx]^{n+1}}{bm} - \frac{n+1}{m} \int \text{Sinh}[a+bx]^{m-2} \text{Tanh}[a+bx]^{n+2} dx$$

■ **Program code:**

```
Int[Sinh[a_.+b_.*x_]^m_.*Tanh[a_.+b_.*x_]^n_,x_Symbol] :=
  Sinh[a+b*x]^m*Tanh[a+b*x]^(n+1)/(b*m) -
  Dist[(n+1)/m,Int[Sinh[a+b*x]^(m-2)*Tanh[a+b*x]^(n+2),x]] /;
FreeQ[{a,b},x] && RationalQ[{m,n}] && m>1 && n<-1
```

■ **Reference:** G&R 2.411.2, CRC 567b

```
Int[Cosh[a_.+b_.*x_]^m_.*Coth[a_.+b_.*x_]^n_,x_Symbol] :=
  Cosh[a+b*x]^m*Coth[a+b*x]^(n+1)/(b*m) +
  Dist[(n+1)/m,Int[Cosh[a+b*x]^(m-2)*Coth[a+b*x]^(n+2),x]] /;
FreeQ[{a,b},x] && RationalQ[{m,n}] && m>1 && n<-1
```

- Reference: G&R 2.411.6, CRC 568b

- Rule: If  $m < -1 \wedge n > 1$ , then

$$\int \sinh[a + bx]^m \tanh[a + bx]^n dx \rightarrow \frac{\sinh[a + bx]^{m+2} \tanh[a + bx]^{n-1}}{b(n-1)} - \frac{m+2}{n-1} \int \sinh[a + bx]^{m+2} \tanh[a + bx]^{n-2} dx$$

- Program code:

```
Int[Sinh[a_.+b_.*x_]^m_*Tanh[a_.+b_.*x_]^n_,x_Symbol] :=
  Sinh[a+b*x]^(m+2)*Tanh[a+b*x]^(n-1)/(b*(n-1)) -
  Dist[(m+2)/(n-1),Int[Sinh[a+b*x]^(m+2)*Tanh[a+b*x]^(n-2),x]] /;
FreeQ[{a,b},x] && RationalQ[{m,n}] && m<-1 && n>1
```

- Reference: G&R 2.411.5, CRC 568a

```
Int[Cosh[a_.+b_.*x_]^m_*Coth[a_.+b_.*x_]^n_,x_Symbol] :=
  -Cosh[a+b*x]^(m+2)*Coth[a+b*x]^(n-1)/(b*(n-1)) +
  Dist[(m+2)/(n-1),Int[Cosh[a+b*x]^(m+2)*Coth[a+b*x]^(n-2),x]] /;
FreeQ[{a,b},x] && RationalQ[{m,n}] && m<-1 && n>1
```

- Reference: G&R 2.411.2, CRC 567b

- Rule: If  $m > 1$ , then

$$\int \sinh[a + bx]^m \tanh[a + bx]^n dx \rightarrow \frac{\sinh[a + bx]^m \tanh[a + bx]^{n-1}}{bm} - \frac{m+n-1}{m} \int \sinh[a + bx]^{m-2} \tanh[a + bx]^n dx$$

- Program code:

```
Int[Sinh[a_.+b_.*x_]^m_*Tanh[a_.+b_.*x_]^n_,x_Symbol] :=
  Sinh[a+b*x]^m*Tanh[a+b*x]^(n-1)/(b*m) -
  Dist[(m+n-1)/m,Int[Sinh[a+b*x]^(m-2)*Tanh[a+b*x]^n,x]] /;
FreeQ[{a,b,n},x] && RationalQ[m] && m>1
```

- Reference: G&R 2.411.1, CRC 567a

```
Int[Cosh[a_.+b_.*x_]^m_*Coth[a_.+b_.*x_]^n_,x_Symbol] :=
  Cosh[a+b*x]^m*Coth[a+b*x]^(n-1)/(b*m) +
  Dist[(m+n-1)/m,Int[Cosh[a+b*x]^(m-2)*Coth[a+b*x]^n,x]] /;
FreeQ[{a,b,n},x] && RationalQ[m] && m>1
```

■ Reference: G&R 2.411.3

■ Rule: If  $n > 1$ , then

$$\int \sinh[a + bx]^m \tanh[a + bx]^n dx \rightarrow -\frac{\sinh[a + bx]^m \tanh[a + bx]^{n-1}}{b(n-1)} + \frac{m+n-1}{n-1} \int \sinh[a + bx]^m \tanh[a + bx]^{n-2} dx$$

■ Program code:

```
Int[Sinh[a_+b_.*x_]^m_.*Tanh[a_+b_.*x_]^n_,x_Symbol] :=
  -Sinh[a+b*x]^m*Tanh[a+b*x]^(n-1)/(b*(n-1)) +
  Dist[(m+n-1)/(n-1),Int[Sinh[a+b*x]^m*Tanh[a+b*x]^(n-2),x]] /;
FreeQ[{a,b,m},x] && RationalQ[n] && n>1
```

■ Reference: G&R 2.411.4

```
Int[Cosh[a_+b_.*x_]^m_.*Coth[a_+b_.*x_]^n_,x_Symbol] :=
  -Cosh[a+b*x]^m*Coth[a+b*x]^(n-1)/(b*(n-1)) +
  Dist[(m+n-1)/(n-1),Int[Cosh[a+b*x]^m*Coth[a+b*x]^(n-2),x]] /;
FreeQ[{a,b,m},x] && RationalQ[n] && n>1
```

■ Reference: G&R 2.411.5, CRC 568a

■ Rule: If  $m < -1 \wedge m+n+1 \neq 0$ , then

$$\int \sinh[a + bx]^m \tanh[a + bx]^n dx \rightarrow \frac{\sinh[a + bx]^{m+2} \tanh[a + bx]^{n-1}}{b(m+n+1)} - \frac{m+2}{m+n+1} \int \sinh[a + bx]^{m+2} \tanh[a + bx]^n dx$$

■ Program code:

```
Int[Sinh[a_+b_.*x_]^m_.*Tanh[a_+b_.*x_]^n_,x_Symbol] :=
  Sinh[a+b*x]^(m+2)*Tanh[a+b*x]^(n-1)/(b*(m+n+1)) -
  Dist[(m+2)/(m+n+1),Int[Sinh[a+b*x]^(m+2)*Tanh[a+b*x]^n,x]] /;
FreeQ[{a,b,n},x] && RationalQ[m] && m<-1 && NonzeroQ[m+n+1]
```

■ Reference: G&R 2.411.6, CRC 568b

```
Int[Cosh[a_+b_.*x_]^m_.*Coth[a_+b_.*x_]^n_,x_Symbol] :=
  -Cosh[a+b*x]^(m+2)*Coth[a+b*x]^(n-1)/(b*(m+n+1)) +
  Dist[(m+2)/(m+n+1),Int[Cosh[a+b*x]^(m+2)*Coth[a+b*x]^n,x]] /;
FreeQ[{a,b,n},x] && RationalQ[m] && m<-1 && NonzeroQ[m+n+1]
```

■ Reference: G&R 2.411.4

■ Rule: If  $n < -1 \wedge m + n + 1 \neq 0$ , then

$$\int \sinh[a + b x]^m \tanh[a + b x]^n dx \rightarrow \frac{\sinh[a + b x]^m \tanh[a + b x]^{n+1}}{b (m + n + 1)} + \frac{n + 1}{m + n + 1} \int \sinh[a + b x]^m \tanh[a + b x]^{n+2} dx$$

■ Program code:

```
Int[Sinh[a_.+b_.*x_]^m_.*Tanh[a_.+b_.*x_]^n_,x_Symbol]:=
  Sinh[a+b*x]^m*Tanh[a+b*x]^(n+1)/(b*(m+n+1)) +
  Dist[(n+1)/(m+n+1),Int[Sinh[a+b*x]^m*Tanh[a+b*x]^(n+2),x]] /;
FreeQ[{a,b,m},x] && RationalQ[n] && n<-1 && NonzeroQ[m+n+1]
```

■ Reference: G&R 2.411.3

```
Int[Cosh[a_.+b_.*x_]^m_.*Coth[a_.+b_.*x_]^n_,x_Symbol] :=
  Cosh[a+b*x]^m*Coth[a+b*x]^(n+1)/(b*(m+n+1)) +
  Dist[(n+1)/(m+n+1),Int[Cosh[a+b*x]^m*Coth[a+b*x]^(n+2),x]] /;
FreeQ[{a,b,m},x] && RationalQ[n] && n<-1 && NonzeroQ[m+n+1]
```

$$\int u \sinh[v] \operatorname{Hyper}[w] \, dx$$

- **Derivation:** Algebraic expansion

- **Basis:**  $\sinh[v] \cosh[w] = \frac{1}{2} \sinh[v+w] + \frac{1}{2} \sinh[v-w]$

- **Rule:** If  $v, w \in \mathbb{P}x \wedge v+w \neq 0 \wedge v-w \neq 0$ , then

$$\int u \sinh[v] \cosh[w] \, dx \rightarrow \frac{1}{2} \int u \sinh[v+w] \, dx + \frac{1}{2} \int u \sinh[v-w] \, dx$$

- **Program code:**

```
Int[u_.*Sinh[v_]*Cosh[w_],x_Symbol] :=
  Dist[1/2,Int[u*Regularize[Sinh[v+w],x],x]] +
  Dist[1/2,Int[u*Regularize[Sinh[v-w],x],x]] /;
(PolynomialQ[v,x] && PolynomialQ[w,x] || IndependentQ[Cancel[v/w],x]) && NonzeroQ[v+w] && NonzeroQ[v-w]
```

- **Derivation:** Algebraic expansion

- **Basis:**  $\sinh[v] \sinh[w] = \frac{1}{2} \cosh[v+w] - \frac{1}{2} \cosh[v-w]$

- **Rule:** If  $v, w \in \mathbb{P}x \wedge v+w \neq 0 \wedge v-w \neq 0$ , then

$$\int u \sinh[v] \sinh[w] \, dx \rightarrow \frac{1}{2} \int u \cosh[v+w] \, dx - \frac{1}{2} \int u \cosh[v-w] \, dx$$

- **Program code:**

```
Int[u_.*Sinh[v_]*Sinh[w_],x_Symbol] :=
  Dist[1/2,Int[u*Regularize[Cosh[v+w],x],x]] -
  Dist[1/2,Int[u*Regularize[Cosh[v-w],x],x]] /;
(PolynomialQ[v,x] && PolynomialQ[w,x] || IndependentQ[Cancel[v/w],x]) && NonzeroQ[v+w] && NonzeroQ[v-w]
```

- **Basis:**  $\cosh[v] \cosh[w] = \frac{1}{2} \cosh[v-w] + \frac{1}{2} \cosh[v+w]$

```
Int[u_.*Cosh[v_]*Cosh[w_],x_Symbol] :=
  Dist[1/2,Int[u*Regularize[Cosh[v-w],x],x]] +
  Dist[1/2,Int[u*Regularize[Cosh[v+w],x],x]] /;
(PolynomialQ[v,x] && PolynomialQ[w,x] || IndependentQ[Cancel[v/w],x]) && NonzeroQ[v+w] && NonzeroQ[v-w]
```

- **Derivation: Algebraic expansion**

- **Basis:**  $\sinh[v] \tanh[w] = \cosh[v] - \cosh[v-w] \operatorname{sech}[w]$

- **Rule:** If  $n > 0 \wedge x \notin v-w \neq 0$ , then

$$\int u \sinh[v] \tanh[w]^n dx \rightarrow \int u \cosh[v] \tanh[w]^{n-1} dx - \cosh[v-w] \int u \operatorname{sech}[w] \tanh[w]^{n-1} dx$$

- **Program code:**

```
Int[u_.*Sinh[v_]*Tanh[w_]^n_.,x_Symbol] :=
  Int[u*Cosh[v]*Tanh[w]^(n-1),x] - Cosh[v-w]*Int[u*Sech[w]*Tanh[w]^(n-1),x] /;
RationalQ[n] && n>0 && FreeQ[v-w,x] && NonzeroQ[v-w]
```

- **Basis:**  $\cosh[v] \coth[w] = \sinh[v] + \cosh[v-w] \operatorname{csch}[w]$

```
Int[u_.*Cosh[v_]*Coth[w_]^n_.,x_Symbol] :=
  Int[u*Sinh[v]*Coth[w]^(n-1),x] + Cosh[v-w]*Int[u*Csch[w]*Coth[w]^(n-1),x] /;
RationalQ[n] && n>0 && FreeQ[v-w,x] && NonzeroQ[v-w]
```

- **Derivation: Algebraic expansion**

- **Basis:**  $\sinh[v] \coth[w] = \cosh[v] + \sinh[v-w] \operatorname{csch}[w]$

- **Rule:** If  $n > 0 \wedge x \notin v-w \neq 0$ , then

$$\int u \sinh[v] \coth[w]^n dx \rightarrow \int u \cosh[v] \coth[w]^{n-1} dx + \sinh[v-w] \int u \operatorname{csch}[w] \coth[w]^{n-1} dx$$

- **Program code:**

```
Int[u_.*Sinh[v_]*Coth[w_]^n_.,x_Symbol] :=
  Int[u*Cosh[v]*Coth[w]^(n-1),x] + Sinh[v-w]*Int[u*Csch[w]*Coth[w]^(n-1),x] /;
RationalQ[n] && n>0 && FreeQ[v-w,x] && NonzeroQ[v-w]
```

- **Basis:**  $\cosh[v] \tanh[w] = \sinh[v] - \sinh[v-w] \operatorname{sech}[w]$

```
Int[u_.*Cosh[v_]*Tanh[w_]^n_.,x_Symbol] :=
  Int[u*Sinh[v]*Tanh[w]^(n-1),x] - Sinh[v-w]*Int[u*Sech[w]*Tanh[w]^(n-1),x] /;
RationalQ[n] && n>0 && FreeQ[v-w,x] && NonzeroQ[v-w]
```



- **Derivation: Algebraic expansion**

- **Basis:**  $\text{Sinh}[v] \text{Sech}[w] = \text{Cosh}[v-w] \text{Tanh}[w] + \text{Sinh}[v-w]$

- **Rule:** If  $n > 0 \wedge x \notin v-w \neq 0$ , then

$$\int u \text{Sinh}[v] \text{Sech}[w]^n dx \rightarrow \text{Cosh}[v-w] \int u \text{Tanh}[w] \text{Sech}[w]^{n-1} dx + \text{Sinh}[v-w] \int u \text{Sech}[w]^{n-1} dx$$

- **Program code:**

```
Int[u_.*Sinh[v_]*Sech[w_]^n_.,x_Symbol] :=
  Cosh[v-w]*Int[u*Tanh[w]*Sech[w]^(n-1),x] + Sinh[v-w]*Int[u*Sech[w]^(n-1),x] /;
RationalQ[n] && n>0 && FreeQ[v-w,x] && NonzeroQ[v-w]
```

- **Basis:**  $\text{Cosh}[v] * \text{Csch}[w] = \text{Cosh}[v-w] * \text{Coth}[w] + \text{Sinh}[v-w]$

```
Int[u_.*Cosh[v_]*Csch[w_]^n_.,x_Symbol] :=
  Cosh[v-w]*Int[u*Coth[w]*Csch[w]^(n-1),x] + Sinh[v-w]*Int[u*Csch[w]^(n-1),x] /;
RationalQ[n] && n>0 && FreeQ[v-w,x] && NonzeroQ[v-w]
```

- **Derivation: Algebraic expansion**

- **Basis:**  $\text{Sinh}[v] \text{Csch}[w] = \text{Sinh}[v-w] \text{Coth}[w] + \text{Cosh}[v-w]$

- **Rule:** If  $n > 0 \wedge x \notin v-w \neq 0$ , then

$$\int u \text{Sinh}[v] \text{Csch}[w]^n dx \rightarrow \text{Sinh}[v-w] \int u \text{Coth}[w] \text{Csch}[w]^{n-1} dx + \text{Cosh}[v-w] \int u \text{Csch}[w]^{n-1} dx$$

- **Program code:**

```
Int[u_.*Sinh[v_]*Csch[w_]^n_.,x_Symbol] :=
  Sinh[v-w]*Int[u*Coth[w]*Csch[w]^(n-1),x] + Cosh[v-w]*Int[u*Csch[w]^(n-1),x] /;
RationalQ[n] && n>0 && FreeQ[v-w,x] && NonzeroQ[v-w]
```

- **Basis:**  $\text{Cosh}[v] \text{Sech}[w] = \text{Sinh}[v-w] \text{Tanh}[w] + \text{Cosh}[v-w]$

```
Int[u_.*Cosh[v_]*Sech[w_]^n_.,x_Symbol] :=
  Sinh[v-w]*Int[u*Tanh[w]*Sech[w]^(n-1),x] + Cosh[v-w]*Int[u*Sech[w]^(n-1),x] /;
RationalQ[n] && n>0 && FreeQ[v-w,x] && NonzeroQ[v-w]
```

$$\int x^m \sinh[a + b x^n]^p \cosh[a + b x^n] dx$$

■ Reference: G&R 2.479.6

■ Rule: If  $m, n, p \in \mathbb{Z} \wedge p \neq -1 \wedge 0 < n \leq m$ , then

$$\int x^m \sinh[a + b x^n]^p \cosh[a + b x^n] dx \rightarrow \frac{x^{m-n+1} \sinh[a + b x^n]^{p+1}}{b n (p+1)} - \frac{m-n+1}{b n (p+1)} \int x^{m-n} \sinh[a + b x^n]^{p+1} dx$$

■ Program code:

```
Int[x_^m_.*Sinh[a_+b_*x_^n_]^p_.*Cosh[a_+b_*x_^n_],x_Symbol] :=
  x^(m-n+1)*Sinh[a+b*x^n]^(p+1)/(b*n*(p+1)) -
  Dist[(m-n+1)/(b*n*(p+1)),Int[x^(m-n)*Sinh[a+b*x^n]^(p+1),x]] /;
FreeQ[{a,b},x] && IntegersQ[m,n,p] && p≠-1 && 0<n≤m
```

■ Reference: G&R 2.479.3

```
Int[x_^m_.*Cosh[a_+b_*x_^n_]^p_.*Sinh[a_+b_*x_^n_],x_Symbol] :=
  x^(m-n+1)*Cosh[a+b*x^n]^(p+1)/(b*n*(p+1)) -
  Dist[(m-n+1)/(b*n*(p+1)),Int[x^(m-n)*Cosh[a+b*x^n]^(p+1),x]] /;
FreeQ[{a,b},x] && IntegersQ[m,n,p] && p≠-1 && 0<n≤m
```

$$\int \sinh[a + b x]^m \cosh[a + b x]^n dx$$

- Reference: G&R 2.411.5, CRC 568a, A&S 4.5.86a with  $m + n + 2 = 0$

- Rule: If  $m + n + 2 = 0 \wedge m + 1 \neq 0$ , then

$$\int \sinh[a + b x]^m \cosh[a + b x]^n dx \rightarrow \frac{\sinh[a + b x]^{m+1} \cosh[a + b x]^{n+1}}{b (m + 1)}$$

- Program code:

```
Int[Sinh[a_.+b_.*x_]^m_.*Cosh[a_.+b_.*x_]^n_,x_Symbol] :=
  Sinh[a+b*x]^(m+1)*Cosh[a+b*x]^(n+1)/(b*(m+1)) /;
FreeQ[{a,b,m,n},x] && ZeroQ[m+n+2] && NonzeroQ[m+1]
```

- Derivation: Integration by substitution

- Basis: If  $\frac{n-1}{2} \in \mathbb{Z}$ , then  $\sinh[z]^m \cosh[z]^n = \sinh[z]^m (1 + \sinh[z]^2)^{\frac{n-1}{2}} \sinh'[z]$

- Note: This rule is used for odd  $n$  since it requires fewer steps and results in a simpler antiderivative than the other rules in this section.

- Rule: If  $\frac{n-1}{2} \in \mathbb{Z} \wedge \neg \left( \frac{m-1}{2} \in \mathbb{Z} \wedge 0 < m < n \right)$ , then

$$\int \sinh[a + b x]^m \cosh[a + b x]^n dx \rightarrow \frac{1}{b} \text{Subst} \left[ \int x^m (1 + x^2)^{\frac{n-1}{2}} dx, x, \sinh[a + b x] \right]$$

- Program code:

```
Int[Sinh[a_.+b_.*x_]^m_.*Cosh[a_.+b_.*x_]^n_,x_Symbol] :=
  Dist[1/b,Subst[Int[Regularize[x^m*(1+x^2)^( (n-1)/2 ),x],x],x,Sinh[a+b*x]] /;
FreeQ[{a,b,m},x] && OddQ[n] && Not[OddQ[m] && 0<m<n]
```

- Basis: If  $\frac{m-1}{2} \in \mathbb{Z}$ , then  $\sinh[z]^m \cosh[z]^n = \cosh[z]^n (-1 + \cosh[z]^2)^{\frac{m-1}{2}} \cosh'[z]$

```
Int[Sinh[a_.+b_.*x_]^m_.*Cosh[a_.+b_.*x_]^n_,x_Symbol] :=
  Dist[1/b,Subst[Int[Regularize[x^n*(-1+x^2)^( (m-1)/2 ),x],x],x,Cosh[a+b*x]] /;
FreeQ[{a,b,n},x] && OddQ[m] && Not[OddQ[n] && 0<n<=m]
```

■ Reference: G&R 2.411.3

■ Rule: If  $m > 1 \wedge n < -1$ , then

$$\int \sinh[a + bx]^m \cosh[a + bx]^n dx \rightarrow \frac{\sinh[a + bx]^{m-1} \cosh[a + bx]^{n+1}}{b(n+1)} - \frac{m-1}{n+1} \int \sinh[a + bx]^{m-2} \cosh[a + bx]^{n+2} dx$$

■ Program code:

```
Int[Sinh[a_.+b_.*x_]^m_*Cosh[a_.+b_.*x_]^n_,x_Symbol] :=
  Sinh[a+b*x]^(m-1)*Cosh[a+b*x]^(n+1)/(b*(n+1)) -
  Dist[(m-1)/(n+1),Int[Sinh[a+b*x]^(m-2)*Cosh[a+b*x]^(n+2),x]] /;
FreeQ[{a,b},x] && RationalQ[{m,n}] && m>1 && n<-1
```

■ Reference: G&R 2.411.4

```
Int[Sinh[a_.+b_.*x_]^m_*Cosh[a_.+b_.*x_]^n_,x_Symbol] :=
  Sinh[a+b*x]^(m+1)*Cosh[a+b*x]^(n-1)/(b*(m+1)) -
  Dist[(n-1)/(m+1),Int[Sinh[a+b*x]^(m+2)*Cosh[a+b*x]^(n-2),x]] /;
FreeQ[{a,b},x] && RationalQ[{m,n}] && m<-1 && n>1
```

■ Reference: G&R 2.411.2, CRC 567b, A&S 4.5.85b

■ Rule: If  $m > 1 \wedge \frac{m-1}{2} \notin \mathbb{Z} \wedge m+n \neq 0 \wedge \neg \left( \frac{n-1}{2} \in \mathbb{Z} \wedge n > 1 \right)$ , then

$$\int \sinh[a + bx]^m \cosh[a + bx]^n dx \rightarrow \frac{\sinh[a + bx]^{m-1} \cosh[a + bx]^{n+1}}{b(m+n)} - \frac{m-1}{m+n} \int \sinh[a + bx]^{m-2} \cosh[a + bx]^n dx$$

■ Program code:

```
Int[Sinh[a_.+b_.*x_]^m_*Cosh[a_.+b_.*x_]^n_,x_Symbol] :=
  Sinh[a+b*x]^(m-1)*Cosh[a+b*x]^(n+1)/(b*(m+n)) -
  Dist[(m-1)/(m+n),Int[Sinh[a+b*x]^(m-2)*Cosh[a+b*x]^n,x]] /;
FreeQ[{a,b,n},x] && RationalQ[m] && m>1 && NonzeroQ[m+n]
```

■ Reference: G&R 2.411.1, CRC 567a, A&S 4.5.85a

```
Int[Sinh[a_.+b_.*x_]^m_*Cosh[a_.+b_.*x_]^n_,x_Symbol] :=
  Sinh[a+b*x]^(m+1)*Cosh[a+b*x]^(n-1)/(b*(m+n)) +
  Dist[(n-1)/(m+n),Int[Sinh[a+b*x]^m*Cosh[a+b*x]^(n-2),x]] /;
FreeQ[{a,b,m},x] && RationalQ[n] && n>1 && NonzeroQ[m+n]
```

- Reference: G&R 2.411.5, CRC 568a, A&S 4.5.86a

- Rule: If  $m < -1 \wedge m + n + 2 \neq 0$ , then

$$\int \sinh[a + b x]^m \cosh[a + b x]^n dx \rightarrow \frac{\sinh[a + b x]^{m+1} \cosh[a + b x]^{n+1}}{b(m+1)} - \frac{m+n+2}{m+1} \int \sinh[a + b x]^{m+2} \cosh[a + b x]^n dx$$

- Program code:

```
Int[Sinh[a_.+b_.*x_]^m_*Cosh[a_.+b_.*x_]^n_,x_Symbol] :=
  Sinh[a+b*x]^(m+1)*Cosh[a+b*x]^(n+1)/(b*(m+1)) -
  Dist[(m+n+2)/(m+1),Int[Sinh[a+b*x]^(m+2)*Cosh[a+b*x]^n,x]] /;
FreeQ[{a,b,n},x] && RationalQ[m] && m<-1 && NonzeroQ[m+n+2]
```

- Reference: G&R 2.411.6, CRC 568b, A&S 4.5.86b

```
Int[Sinh[a_.+b_.*x_]^m_*Cosh[a_.+b_.*x_]^n_,x_Symbol] :=
  -Sinh[a+b*x]^(m+1)*Cosh[a+b*x]^(n+1)/(b*(n+1)) +
  Dist[(m+n+2)/(n+1),Int[Sinh[a+b*x]^m*Cosh[a+b*x]^(n+2),x]] /;
FreeQ[{a,b,m},x] && RationalQ[n] && n<-1 && NonzeroQ[m+n+2]
```

- Derivation: Integration by substitution

- Basis: If  $\frac{1}{m} \in \mathbb{Z}$ , then  $\frac{\sinh[z]^m}{\cosh[z]^n} = \frac{\left(\frac{\sinh[z]^m}{\cosh[z]^m}\right)^{1/m}}{m \left(1 - \left(\frac{\sinh[z]^m}{\cosh[z]^m}\right)^{2/m}\right)} \partial_z \frac{\sinh[z]^m}{\cosh[z]^n}$

- Note: This rule should be replaced with a more general one.

- Rule: If  $\frac{1}{m} \in \mathbb{Z} \wedge \frac{1}{m} > 1$ , then

$$\int \frac{\sinh[a + b x]^m}{\cosh[a + b x]^m} dx \rightarrow \frac{1}{b m} \text{Subst} \left[ \int \frac{x^{1/m}}{1 - x^{2/m}} dx, x, \frac{\sinh[a + b x]^m}{\cosh[a + b x]^m} \right]$$

- Program code:

```
Int[Sinh[a_.+b_.*x_]^m_*Cosh[a_.+b_.*x_]^n_,x_Symbol] :=
  Dist[1/(b*m),Subst[Int[x^(1/m)/(1-x^(2/m)),x],x,Sinh[a+b*x]^m/Cosh[a+b*x]^m]] /;
FreeQ[{a,b},x] && ZeroQ[m+n] && IntegerQ[1/m] && 1/m>1
```

- Basis: If  $\frac{1}{n} \in \mathbb{Z}$ , then  $\frac{\cosh[z]^n}{\sinh[z]^n} = \frac{\left(\frac{\cosh[z]^n}{\sinh[z]^n}\right)^{1/n}}{n \left(1 - \left(\frac{\cosh[z]^n}{\sinh[z]^n}\right)^{2/n}\right)} \partial_z \frac{\cosh[z]^n}{\sinh[z]^n}$

```
Int[Sinh[a_.+b_.*x_]^m_*Cosh[a_.+b_.*x_]^n_,x_Symbol] :=
  Dist[1/(b*n),Subst[Int[x^(1/n)/(1-x^(2/n)),x],x,Cosh[a+b*x]^n/Sinh[a+b*x]^n]] /;
FreeQ[{a,b},x] && ZeroQ[m+n] && IntegerQ[1/n] && 1/n>1
```

$$\int x^m \left( a + b \cosh[d + e x]^2 + c \sinh[d + e x]^2 \right)^n dx$$

- **Derivation:** Algebraic simplification

- **Basis:**  $a + b \cosh[z]^2 + c \sinh[z]^2 = \frac{1}{2} (2a + b - c + (b + c) \cosh[2z])$

- **Rule:** If  $m \in \mathbb{Z} \wedge m > 0 \wedge a + b \neq 0 \wedge a + c \neq 0$ , then

$$\int \frac{x^m}{a + b \cosh[d + e x]^2 + c \sinh[d + e x]^2} dx \rightarrow 2 \int \frac{x^m}{2a + b - c + (b + c) \cosh[2d + 2ex]} dx$$

- **Program code:**

```
Int[x^m_/(a_+b_.*Cosh[d_+e_.*x_]^2+c_.*Sinh[d_+e_.*x_]^2),x_Symbol] :=
  Dist[2,Int[x^m/(2*a+b-c+(b+c)*Cosh[2*d+2*e*x]),x]] /;
FreeQ[{a,b,c,d,e},x] && IntegerQ[m] && m>0 && NonzeroQ[a+b] && NonzeroQ[a+c]
```

$$\int x^m (a + b \sinh[c + d x] \cosh[c + d x])^n dx$$

- **Derivation:** Algebraic simplification

- **Basis:**  $\sinh[z] \cosh[z] = \frac{1}{2} \sinh[2 z]$

- **Rule:** If  $m \in \mathbb{Z} \wedge m > 0$ , then

$$\int \frac{x^m}{a + b \sinh[c + d x] \cosh[c + d x]} dx \rightarrow \int \frac{x^m}{a + \frac{1}{2} b \sinh[2 c + 2 d x]} dx$$

- **Program code:**

```
Int[x_^m_/ (a_+b_.*Sinh[c_+d_.*x_]*Cosh[c_+d_.*x_]),x_Symbol] :=
  Int[x^m/(a+b*Sinh[2*c+2*d*x]/2),x] /;
FreeQ[{a,b,c,d},x] && IntegerQ[m] && m>0
```

- **Derivation:** Algebraic simplification

- **Basis:**  $\sinh[z] \cosh[z] = \frac{1}{2} \sinh[2 z]$

- **Rule:** If  $n - \frac{1}{2} \in \mathbb{Z}$ , then

$$\int (a + b \sinh[c + d x] \cosh[c + d x])^n dx \rightarrow \int \left( a + \frac{1}{2} b \sinh[2 c + 2 d x] \right)^n dx$$

- **Program code:**

```
Int[(a_+b_.*Sinh[c_+d_.*x_]*Cosh[c_+d_.*x_])^n_,x_Symbol] :=
  Int[(a+b*Sinh[2*c+2*d*x]/2)^n_,x] /;
FreeQ[{a,b,c,d},x] && HalfIntegerQ[n]
```

$$\int \sinh[a + b x^n]^p \cosh[a + b x^n]^p dx$$

- **Derivation:** Algebraic simplification

- **Basis:**  $\sinh[z] \cosh[z] = \frac{1}{2} \sinh[2z]$

- **Rule:** If  $n, p \in \mathbb{Z}$ , then

$$\int \sinh[a + b x^n]^p \cosh[a + b x^n]^p dx \rightarrow \frac{1}{2} \int \sinh[2a + 2b x^n]^p dx$$

- **Program code:**

```
Int[Sinh[a_.+b_.*x_^n_]^p_.*Cosh[a_.+b_.*x_^n_]^p_.,x_Symbol] :=
  Dist[1/2,Int[Sinh[2*a+2*b*x^n]^p,x]] /;
FreeQ[{a,b},x] && IntegersQ[n,p]
```



$$\int (a \operatorname{Csch}[c + d x] + a \operatorname{Sinh}[c + d x])^n dx$$

■ **Derivation: Algebraic simplification**

■ **Basis:**  $\operatorname{Csch}[z] + \operatorname{Sinh}[z] = \operatorname{Cosh}[z] \operatorname{Coth}[z]$

■ **Rule:**

$$\int (a \operatorname{Csch}[c + d x] + a \operatorname{Sinh}[c + d x])^n dx \rightarrow \int (a \operatorname{Cosh}[c + d x] \operatorname{Coth}[c + d x])^n dx$$

■ **Program code:**

```
Int[(a_.*Csch[c_.+d_.*x_]+b_.*Sinh[c_.+d_.*x_])^n_,x_Symbol] :=
  Int[(a*Cosh[c+d*x]*Coth[c+d*x])^n,x] /;
FreeQ[{a,b,c,d,n},x] && ZeroQ[a-b]
```

```
Int[(a_.*Sech[c_.+d_.*x_]+b_.*Cosh[c_.+d_.*x_])^n_,x_Symbol] :=
  Int[(-a*Sinh[c+d*x]*Tanh[c+d*x])^n,x] /;
FreeQ[{a,b,c,d,n},x] && ZeroQ[a+b]
```

$$\int \operatorname{sech}[v]^m (a + b \operatorname{Tanh}[v])^n dx$$

■ **Derivation:** Algebraic simplification

■ **Basis:**  $\frac{a+b \operatorname{Tanh}[z]}{\operatorname{sech}[z]} = a \operatorname{Cosh}[z] + b \operatorname{Sinh}[z]$

■ **Rule:** If  $m, n \in \mathbb{Z} \wedge m+n=0 \wedge \frac{m-1}{2} \in \mathbb{Z}$ , then

$$\int \operatorname{sech}[v]^m (a + b \operatorname{Tanh}[v])^n dx \rightarrow \int (a \operatorname{Cosh}[v] + b \operatorname{Sinh}[v])^n dx$$

■ **Program code:**

```
Int[Sech[v_]^m_.*(a_+b_.*Tanh[v_])^n_., x_Symbol] :=
  Int[(a*Cosh[v]+b*Sinh[v])^n,x] /;
FreeQ[{a,b},x] && IntegersQ[m,n] && m+n==0 && OddQ[m]
```

```
Int[Csch[v_]^m_.*(a_+b_.*Coth[v_])^n_., x_Symbol] :=
  Int[(b*Cosh[v]+a*Sinh[v])^n,x] /;
FreeQ[{a,b},x] && IntegersQ[m,n] && m+n==0 && OddQ[m]
```

$$\int x^m \operatorname{Csch}[a + b x]^n \operatorname{Sech}[a + b x]^p dx$$

- **Derivation:** Algebraic simplification

- **Basis:**  $\operatorname{Csch}[z] \operatorname{Sech}[z] = 2 \operatorname{Csch}[2z]$

- **Rule:** If  $n \in \mathbb{Z}$ , then

$$\int x^m \operatorname{Csch}[a + b x]^n \operatorname{Sech}[a + b x]^n dx \rightarrow 2^n \int x^m \operatorname{Csch}[2a + 2bx]^n dx$$

- **Program code:**

```
Int[x_^m_.*Csch[a_.+b_.*x_]^n_.*Sech[a_.+b_.*x_]^n_. , x_Symbol] :=
  Dist[2^n, Int[x^m*Csch[2*a+2*b*x]^n, x]] /;
FreeQ[{a,b}, x] && RationalQ[m] && IntegerQ[n]
```

- **Derivation:** Integration by parts

- **Rule:** If  $n, p \in \mathbb{Z} \wedge m > 0 \wedge n \neq p$ , then

$$\int x^m \operatorname{Csch}[a + b x]^n \operatorname{Sech}[a + b x]^p dx \rightarrow x^m \int \operatorname{Csch}[a + b x]^n \operatorname{Sech}[a + b x]^p dx - m \int x^{m-1} \left( \int \operatorname{Csch}[a + b x]^n \operatorname{Sech}[a + b x]^p dx \right) dx$$

- **Program code:**

```
Int[x_^m_.*Csch[a_.+b_.*x_]^n_.*Sech[a_.+b_.*x_]^p_. , x_Symbol] :=
  Module[{u=Block[{ShowSteps=False, StepCounter=None}, Int[Csch[a+b*x]^n*Sech[a+b*x]^p, x]]},
    x^m*u - Dist[m, Int[x^(m-1)*u, x]] /;
FreeQ[{a,b}, x] && RationalQ[m] && IntegersQ[n,p] && m>0 && n≠p
```

$$\int u \left( a \operatorname{Tanh}[v]^m + b \operatorname{Sech}[v]^m \right)^n dx$$

■ **Derivation:** Algebraic simplification

■ **Basis:** If  $a^2 + b^2 = 0$ , then  $a \operatorname{Tanh}[z] + b \operatorname{Sech}[z] = a \operatorname{Tanh}\left[\frac{z}{2} - \frac{\pi}{4} \frac{a}{b}\right]$

■ **Rule:** If  $a^2 - b^2 = 0 \bigwedge \frac{n}{2} \in \mathbb{Z}$ , then

$$\int (a \operatorname{Tanh}[v] + b \operatorname{Sech}[v])^n dx \rightarrow a^n \int \operatorname{Tanh}\left[\frac{v}{2} - \frac{\pi}{4} \frac{a}{b}\right]^n dx$$

■ **Program code:**

```
Int[(a_.*Tanh[v_]+b_.*Sech[v_])^n_,x_Symbol] :=
  Dist[a^n,Int[Tanh[v/2-Pi/4*a/b]^n,x]] /;
FreeQ[{a,b},x] && ZeroQ[a^2+b^2] && EvenQ[n]
```

■ **Basis:** If  $a^2 - b^2 = 0$ , then  $a \operatorname{Coth}[z] + b \operatorname{Csch}[z] = a \operatorname{Coth}\left[\frac{z}{2} + \frac{\pi}{4} \frac{a-b}{b}\right]$

```
Int[(a_.*Coth[v_]+b_.*Csch[v_])^n_,x_Symbol] :=
  Dist[a^n,Int[Coth[v/2+Pi*I/4*(a-b)/b]^n,x]] /;
FreeQ[{a,b},x] && ZeroQ[a^2-b^2] && EvenQ[n]
```

■ **Derivation:** Algebraic simplification

■ **Basis:**  $a \operatorname{Sech}[z] + b \operatorname{Tanh}[z] = \frac{a+b \sinh[z]}{\cosh[z]}$

■ **Rule:** If  $m, n \in \mathbb{Z} \bigwedge \left(\frac{mn-1}{2} \in \mathbb{Z} \bigvee mn < 0\right) \bigwedge \neg (m = 2 \wedge a - b = 0)$ , then

$$\int u \left( a \operatorname{Tanh}[v]^m + b \operatorname{Sech}[v]^m \right)^n dx \rightarrow \int \frac{u \left( a + b \sinh[v]^m \right)^n}{\cosh[v]^{mn}} dx$$

■ **Program code:**

```
Int[u_.*(a_.*Sech[v_]^m_.+b_.*Tanh[v_]^m_.)^n_,x_Symbol] :=
  Int[u*(a+b*Sinh[v]^m)^n/Cosh[v]^(m*n),x] /;
FreeQ[{a,b},x] && IntegersQ[m,n] && (OddQ[m*n] || m*n<0) && Not[m==2 && ZeroQ[a-b]]
```

```
Int[u_.*(a_.*Csch[v_]^m_.+b_.*Coth[v_]^m_.)^n_,x_Symbol] :=
  Int[u*(a+b*Cosh[v]^m)^n/Sinh[v]^(m*n),x] /;
FreeQ[{a,b},x] && IntegersQ[m,n] && (OddQ[m*n] || m*n<0) && Not[m==2 && ZeroQ[a+b]]
```